What is a Bianchi modular form? And why does anyone care?

Y-RANT VI

Lewis Combes

Classical modular forms

Write $\mathcal{H}_2 = \{x + iy \mid y > 0\}$. A **classical modular form** is a function $f : \mathcal{H}_2 \to \mathbb{C}$ satisfying

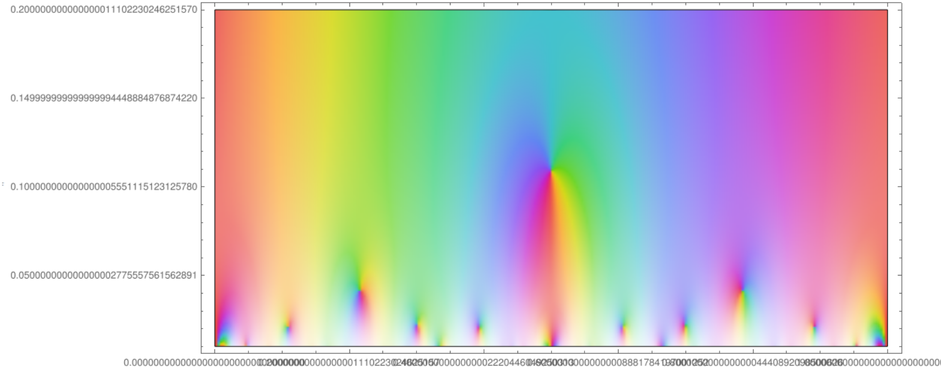
1. $f(z)$ is holomorphic;
2. $f(\frac{az+b}{cz+d}) = (cz + d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$;
3. $f(z)$ is "holomorphic at the cusps" (boundedness)

The group Γ is called the **level**, and k is called the **weight**.

A classical modular form

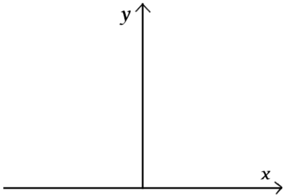
Let $f(z) = \dfrac{\eta(2z)^2\eta(3z)^2\eta(14z)^2\eta(21z)^2}{\eta(z)\eta(6z)\eta(7z)\eta(42z)} - \dfrac{\eta(z)^2\eta(6z)^2\eta(7z)^2\eta(42z)^2}{\eta(2z)\eta(3z)\eta(14z)\eta(21z)}.$

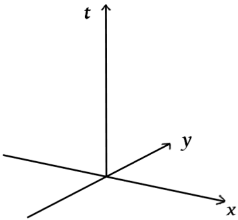
This is the cusp form with LMFDB label `42.2.a.a`.



How to think about Bianchi modular forms I

$$\begin{array}{rcl} \mathbb{Q} & \rightsquigarrow & K \\ \mathbb{Z} & \rightsquigarrow & \mathbb{Z}_K \\ \mathrm{SL}_2(\mathbb{R}) & \rightsquigarrow & \mathrm{SL}_2(\mathbb{C}) \\ \mathrm{SO}(2) & \rightsquigarrow & \mathrm{SU}(2) \\ \mathcal{H}_2 & \rightsquigarrow & \mathcal{H}_3 \\ f : \mathcal{H}_2 \to \mathbb{C} & \rightsquigarrow & F : \mathcal{H}_3 \to \mathbb{C}^r \end{array}$$





How to think about Bianchi modular forms II

Geoffrey Hinton:

> - If you are not used to thinking about hyper-planes in high-dimensional spaces, now is the time to learn.
> - To deal with hyper-planes in a 14-dimensional space, visualize a 3-D space and say “fourteen” to yourself very loudly. Everyone does it.

Geoffrey Hinton:

> - If you are not used to thinking about hyper-planes in high-dimensional spaces, now is the time to learn.
> - To deal with hyper-planes in a 14-dimensional space, visualize a 3-D space and say "fourteen" to yourself very loudly. Everyone does it.

To deal with Bianchi modular forms, imagine a classical modular form and say "imaginary quadratic field" to yourself very loudly. Everyone does it.

A more formal definition

A **Bianchi modular form** is a function $F\colon \mathcal{H}_3 \to \mathbb{C}^r$ satisfying

1. F is harmonic;
2. F satisfies a transformation property;
3. "boundedness at the cusps"

A more formal definition

A **Bianchi modular form** is a function $F\colon \mathcal{H}_3 \to \mathbb{C}^r$ satisfying

1. F is harmonic;
2. F satisfies a transformation property;
3. "boundedness at the cusps"

Weight 2: Let $\tau = (z,t) \in \mathcal{H}_3$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z}_K)$. Then, writing $r = cz + d$, $s = \overline{c}t$,

$$\begin{pmatrix} F_0(\gamma\tau) \\ F_1(\gamma\tau) \\ F_2(\gamma\tau) \end{pmatrix} = \begin{pmatrix} r^2 & 2rs & s^2 \\ -r\overline{s} & r\overline{r} - s\overline{s} & \overline{r}s \\ \overline{s}^2 & -2\overline{rs} & \overline{r}^2 \end{pmatrix} \begin{pmatrix} F_0(\tau) \\ F_1(\tau) \\ F_2(\tau) \end{pmatrix}$$

up to some "en-nice-ening".

Transformation rule is given by a representation of $\mathrm{SU}(2)$.

Visualising Bianchi modular forms

https://lewismcombes.github.io/pages/misc/bianchi_images/visualising_BMFs.html

Compared to classical modular forms

Bianchi modular forms have

- Fourier (-Whittaker) expansions
- Hecke operators
- newforms & cusp forms
- Galois representations
- Eichler-Shimura isomorphism
- L-series
- modular symbols

But there is still a lot we don't understand.

Open problems I: dimension formula

Since $\mathcal{H}_3$ is 3-dimensional over $\mathbb{R}$, there is no algebraic geometry to exploit. Classical dimension formula comes from Riemann-Roch. **No formulae for**

$$\dim M_k(\Gamma_0(\mathfrak{n})), \quad \dim S_k(\Gamma_0(\mathfrak{n}))$$

are known.

Open problems I: dimension formula

Since $\mathcal{H}_3$ is 3-dimensional over $\mathbb{R}$, there is no algebraic geometry to exploit. Classical dimension formula comes from Riemann-Roch. **No formulae for**

$$\dim M_k(\Gamma_0(\mathfrak{n})), \quad \dim S_k(\Gamma_0(\mathfrak{n}))$$

are known.

This problem is probably quite hard.

Open problem II: connection to elliptic curves

When F is a weight 2 Bianchi cusp form (+the usual criteria), one expects* to find an elliptic curve E such that

$$L(F, s) = L(E, s).$$

No general construction is known.

Open problem II: connection to elliptic curves

When F is a weight 2 Bianchi cusp form (+the usual criteria), one expects* to find an elliptic curve E such that

$$L(F, s) = L(E, s).$$

No general construction is known.

This problem is probably **very** hard.

Open problem III: the torsion phenomenon

Eichler-Shimura: Bianchi forms can be found in $H^1(\Gamma, V)$. Due to torsion, we often have

$$\dim H^1(\Gamma, \mathbb{F}_p) > \dim H^1(\Gamma, \mathbb{C}).$$

i.e. there are mod p modular forms that don't lift to characteristic 0.

In 2015, Scholze proved these classes have their own Galois representations, so they should somewhere into the Langlands program.

1. Some of them lift to characteristic 0 at higher level $\Gamma' \leq \Gamma$, but some seem not to. But nobody knows.
2. Their periods "should" relate to ranks of Selmer groups (a la BSD), but this is also not known.
3. Probably other things too.

Open problem IV: Your favourite property of classical modular forms

Philosophically, things that are true for classical modular forms “ought to be” true for Bianchi too (although some are not).

$\rightsquigarrow$ whatever you know about classical modular forms, it is probably open (and requires actual work to formulate and prove) for Bianchi modular forms.

Why should you care?

1. Important test case for general conjectures (fairly easy to work with, fairly hard to understand)

Why should you care?

1. Important test case for general conjectures (fairly easy to work with, fairly hard to understand)
2. Deveop your understanding of the subtleties of automorphic forms

Why should you care?

1. Important test case for general conjectures (fairly easy to work with, fairly hard to understand)
2. Deveop your understanding of the subtleties of automorphic forms
3. "Easy" to study—any question about classical can be asked about Bianchi

Why should you care?

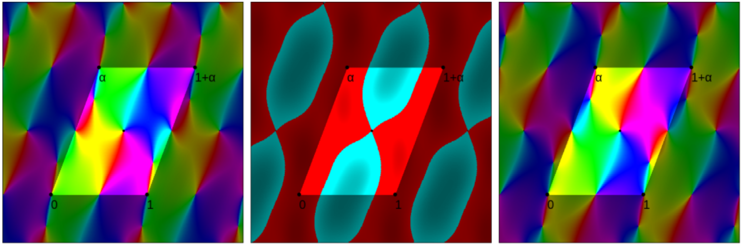
1. Important test case for general conjectures (fairly easy to work with, fairly hard to understand)
2. Deveop your understanding of the subtleties of automorphic forms
3. "Easy" to study—any question about classical can be asked about Bianchi (write papers = get job)

Why should you care?

1. Important test case for general conjectures (fairly easy to work with, fairly hard to understand)
2. Deveop your understanding of the subtleties of automorphic forms
3. "Easy" to study—any question about classical can be asked about Bianchi (write papers = get job)
4. Everyone else is going to study them. You don't want to be left behind, do you?


MISSING!
HAVE YOU SEEN MY ARTHUR PARAMETERS?
IF SO CONTACT: daniel.fretwell@bristol.ac.uk
REWARD: MATHEMATICAL ENLIGHTENMENT (+POTENTIAL PAPER)
Global Arthur parameters
Lecture 5

ATTENTION!



The Bianchi modular form associated to the elliptic curve

$$E: y^2 + xy = x^3 + \alpha x^2 + x + 1, \qquad \alpha = \frac{1+\sqrt{-7}}{2}$$

seems to vanish at the point

$$z = \frac{3+\sqrt{-7}}{4}, \ t = \frac{1}{\sqrt[4]{88}}$$

This is weird, right? If you can prove this, please email lmc577@proton.me
to claim your **reward**!

The word "reward" here refers to the spiritually rewarding *experience* of doing mathematics, and knowing that you have helped advance the theory of Bianchi modular forms, and in no way reflects a promise of remuneration of any kind. That said, if you figure this out I *will* buy you a drink.