Counting prime numbers: the Riemann hypothesis

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Lewis Combes

University of Sheffield

The zeta function

The zeta function $\zeta(s)$ is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for } \mathrm{Re}(s) > 1.$$

It was studied by Euler, who evaluated it for all even integers. The most famous of these is the solution to **the Basel problem**:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Euler gave his first proof in 1734.

Euler's solution to the Basel problem

The goal is to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Euler started with the formula

$$\frac{\sin(x)}{x} = \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{x}{n\pi}\right).$$

In analogy with writing a polynomial

$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

as the product

$$(x - r_1)(x - r_2)\ldots(x - r_n).$$

The Euler product

Euler did more work on the zeta function. In 1737 he proved the **Euler product formula**:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

This was the first time a connection had been found between the zeta function and the primes. The formula follows from the fundamental theorem of arithmetic.

Prime numbers

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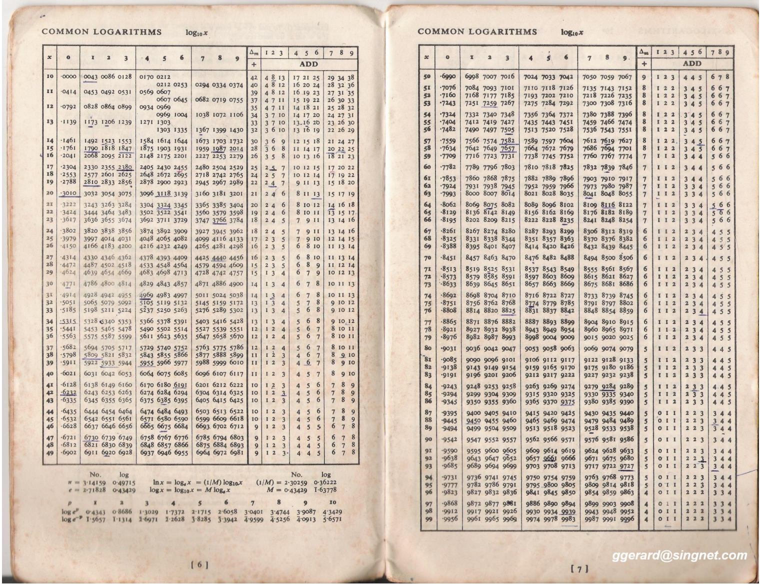
Given a number X, how many primes p are there such that $p \leq X$?

Log tables



COMMON LOGARITHMS $\log_{10} x$

x	0	1	2	3	4	5	6	7	8	9	Δm	1	2	3	4	5	6	7	8	9
											+	ADD								
10	·0000	0043	0086	0128	0170	0212					42	4	8	13	17	21	25	29	34	38
						0212	0253	0294	0334	0374	40	4	8	12	16	20	24	28	32	36
11	·0414	0453	0492	0531	0569	0607					39	4	8	12	16	19	23	27	31	35
						0607	0645	0682	0719	0755	37	4	7	11	15	19	22	26	30	33
12	·0792	0828	0864	0899	0934	0969					35	4	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	34	3	7	10	14	17	20	24	27	31
13	·1139	1173	1206	1239	1271	1303					33	3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	32	3	6	10	13	16	19	22	26	29
14	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30	3	6	9	12	15	18	21	24	27
15	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28	3	6	8	11	14	17	20	22	25
16	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26	3	5	8	10	13	16	18	21	23
17	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25	2	5	7	10	12	15	17	20	22
18	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24	2	5	7	10	12	14	17	19	22
19	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22	2	4	7	9	11	13	15	18	20
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21	2	4	6	8	11	13	15	17	19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20	2	4	6	8	10	12	14	16	18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19	2	4	6	8	10	11	13	15	17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18	2	4	5	7	9	11	13	14	16
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18	2	4	5	7	9	11	13	14	16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17	2	3	5	7	9	10	12	14	15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16	2	3	5	6	8	10	11	13	14
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16	2	3	5	6	8	10	11	13	14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15	2	3	5	6	8	9	11	12	14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15	1	3	4	6	7	9	10	12	13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14	1	3	4	6	7	8	10	11	13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14	1	3	4	6	7	8	10	11	13
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13	1	3	4	5	7	8	9	10	12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13	1	3	4	5	6	8	9	10	12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13	1	3	4	5	6	8	9	10	12
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12	1	2	4	5	6	7	8	10	11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12	1	2	4	5	6	7	8	10	11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12	1	2	4	5	6	7	8	10	11
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11	1	2	3	4	6	7	8	9	10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11	1	2	3	4	6	7	8	9	10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11	1	2	3	4	5	7	8	9	10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10	1	2	3	4	5	6	7	8	9
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10	1	2	3	4	5	6	7	8	9
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10	1	2	3	4	5	6	7	8	9
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10	1	2	3	4	5	6	7	8	9
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10	1	2	3	4	5	6	7	8	9
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9	1	2	3	4	5	5	6	7	8
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9	1	2	3	4	5	5	6	7	8
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9	1	2	3	4	4	5	6	7	8
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9	1	2	3	4	4	5	6	7	8

	No.	log
π =	3·14159	0·49715
e =	2·71828	0·43429

$\ln x = \log_e x = (1/M)\log_{10} x$

$\log x = \log_{10} x = M \log_e x$

	No.	log
$(1/M)$ =	2·30259	0·36222
M =	0·43429	$\bar{1}$·63778

p	1	2	3	4	5	6	7	8	9	10
$\log e^{p}$	0·4343	0·8686	1·3029	1·7372	2·1715	2·6058	3·0401	3·4744	3·9087	4·3429
$\log e^{-p}$	$\bar{1}$·5657	$\bar{1}$·1314	$\bar{2}$·6971	$\bar{2}$·2628	$\bar{3}$·8285	$\bar{3}$·3942	$\bar{4}$·9599	$\bar{4}$·5256	$\bar{4}$·0913	$\bar{5}$·6571

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COMMON LOGARITHMS $\log_{10} x$

x	0	1	2	3	4	5	6	7	8	9	Δm	1	2	3	4	5	6	7	8	9
											+	ADD								
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9	1	2	3	4	4	5	6	7	8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8	1	2	2	3	4	5	6	6	7
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8	1	2	2	3	4	5	6	6	7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8	1	2	2	3	4	5	6	6	7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8	1	2	2	3	4	5	6	6	7
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8	1	2	2	3	4	5	6	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8	1	2	2	3	4	5	6	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8	1	2	2	3	4	5	6	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	8	1	2	2	3	4	5	6	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7	1	1	2	3	4	4	5	6	6
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7	1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7	1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7	1	1	2	3	3	4	5	6	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7	1	1	2	3	3	4	5	6	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7	1	1	2	3	3	4	5	6	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7	1	1	2	3	3	4	5	6	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6	1	1	2	2	3	4	4	5	5
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6	1	1	2	2	3	4	4	5	5
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6	1	1	2	2	3	4	4	5	5
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6	1	1	2	2	3	4	4	5	5
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6	1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6	1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6	1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6	1	1	2	2	3	4	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6	1	1	2	2	3	4	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6	1	1	2	2	3	4	4	5	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6	1	1	2	2	3	4	4	5	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	6	1	1	2	2	3	4	4	5	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5	1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5	1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5	1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5	1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5	1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5	1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5	1	1	2	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5	0	1	1	2	2	3	3	4	4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5	0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5	0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5	0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5	0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5	0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5	0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5	0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5	0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4	0	1	1	2	2	2	3	3	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4	0	1	1	2	2	2	3	3	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4	0	1	1	2	2	2	3	3	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4	0	1	1	2	2	2	3	3	4

ggerard@singnet.com

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Gauss (1777-1885) and the prime number theorem

"Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end I counted the primes in several chiliads and recorded the results on the attached white pages. I soon recognized that **behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm**, *so that the number of primes below a given bound n is approximately equal to*

$$\int \frac{dn}{\log(n)},$$

where the logarithm is understood to be hyperbolic."

Y. Tschinkel. *About the cover: on the distribution of primes—Gauss' tables*. Bull. Amer. Math. Soc. 43 (1) (2005), pp.89-91.

Gauss and the prime numer theorem (cont.)

In modern language, Gauss' observation is written in the following way. Write

$$\pi(X) = \sum_{n \leq X} \mathbb{1}(n \text{ is prime}).$$

Then

$$\pi(X) \approx \frac{X}{\log(X)}.$$

Gauss and the prime numer theorem (cont.)

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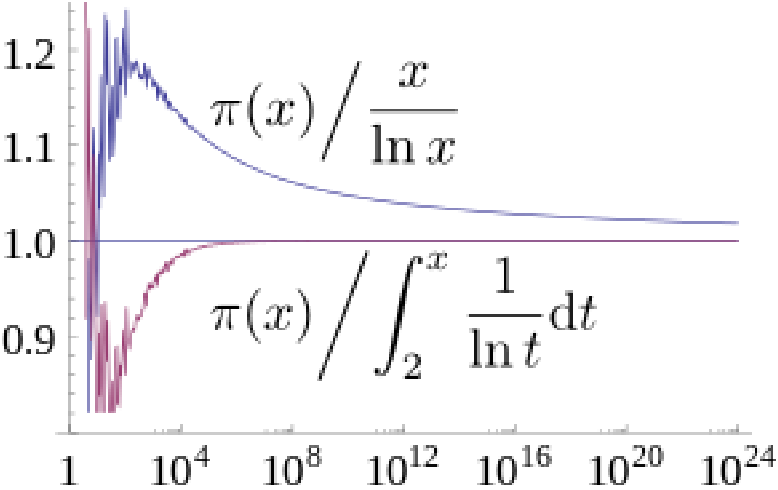
Then

$$\pi(X) \approx \frac{X}{\log(X)}.$$

The statement of the **prime number theorem** is more precise than this. It says that

$$\frac{\pi(X)}{X/\log(X)} \to 1 \quad \text{as } X \to \infty.$$

The prime number theorem



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How to prove the prime number theorem

Attempts to prove PNT led to the creation of modern analytic number theory. The central method of analytic number theory is to estimate error terms. Write

$$\pi(X) = \frac{X}{\log(X)} + E(X),$$

where $E(X) := \pi(X) - \frac{X}{\log(X)}$.

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Showing $E(X)/(X/\log(X)) \to 0$ and $x \to \infty$ proves PNT.

Prime number theorem stats

X	$\pi(X)$	$\frac{X}{\log(X)}$	$E(X) = \pi(X) - \frac{X}{\log(X)}$
10	4	4.3429	-0.3429
100	25	21.714	3.285
1000	168	144.764	23.235
10^4	1229	1085.74	143.26
10^5	9592	8685.89	906.11
10^6	78498	72382.41	6115.59
10^7	664579	620420.69	44158.31
10^8	5761455	542681.02	332773.98
10^9	50847534	48254942.43	2592591.57

Big O notation

We want a nicer way to write error terms.

We say $f(X) = O(g(X))$ if $|f(X)| < C \cdot g(X)$ eventually.

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$$\begin{aligned} X &= O(X) \\ X^2 + X + 1 &= O(X^2) \\ \sin(X) &= O(1) \\ 10\sin(X) &= O(1) \end{aligned}$$

PNT with Big O

The prime number theorem has error term

$$\pi(X) = \frac{X}{\log(X)} + O\left(\frac{X}{\log(X)^2}\right)$$

PNT with Big O

The prime number theorem has error term

$$\pi(X) = \frac{X}{\log(X)} + O\left(\frac{X}{\log(X)^2}\right)$$

In particular,

$$\lim_{X\to\infty} \frac{\pi(X)}{X/\log(X)} \to 1.$$

Riemann's work

Bernhard Riemann (1826-1886) worked mostly in analysis and geometry.

His only contribution to number theory was the paper *"Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"* (in English: *"On the Number of Primes Less Than a Given Magnitude"*).

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In this paper, he extended the domain of convergence of zeta to $\mathbb{C}\setminus\{1\}$ using a technique called **analytic continuation**. He also proved the **functional equation** of the zeta function.

$$\zeta(s) = 2^s \pi^{s-1} \sin(\tfrac{\pi s}{2}) \Gamma(1-s) \zeta(1-s).$$

Here $\Gamma(s)$ the is Gamma function, defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Riemann's work (cont.)

Technically, Riemann computed an exact formula for a related function

$$\pi^*(X) = \sum_{p^k \leq X} \frac{1}{k}.$$

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This counts primes and their powers, assigning lower weight to higher powers. A few simple manipulations can be used to count just the primes using this function.

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This counts primes and their powers, assigning lower weight to higher powers. A few simple manipulations can be used to count just the primes using this function.

The formula is

$$\pi^*(X) = \mathrm{li}(X) - \sum_{\zeta(\rho)=0} \mathrm{li}(X^\rho),$$

where

$$\mathrm{li}(X) = \int_2^X \frac{1}{\log(t)} dt.$$

Zeta reappears

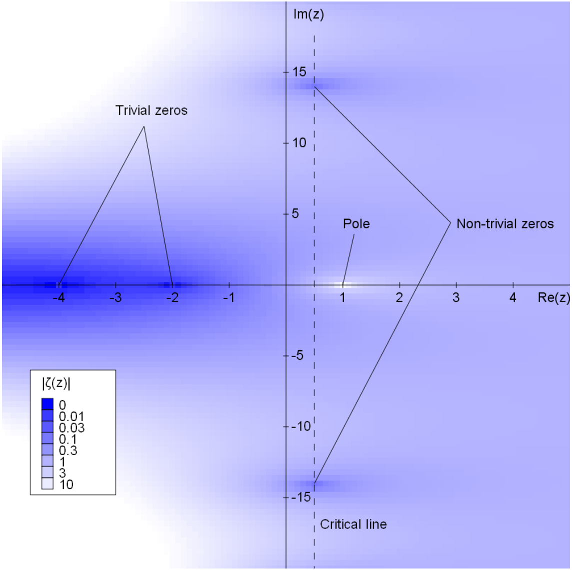
Finally we come back around to the zeta function. The second term in the explicit expression is

$$\sum_{\zeta(\rho)=0} \mathrm{li}(X^\rho).$$

This sum is taken over all zeros $\rho \in \mathbb{C}$ of zeta.

This was the first indication that the zeros of the zeta function are related to the distribution of the primes.

The zeta function (revisited)



Zero-free regions

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The Riemann hypothesis implies the **best possible bound**

$$\pi(X) = \frac{X}{\log(X)} + O(x^{1/2}\log(X)).$$

Riemann's work (cont.)

$$\pi^*(X) = \mathrm{li}(X) - \sum_{\substack{\zeta(\rho)=0 \\ 0<\mathrm{Re}(\rho)<1}} \mathrm{li}(X^\rho) - \log(2) - \int_X^\infty \frac{1}{t(t^2-1)\log(t)}\,dt.$$

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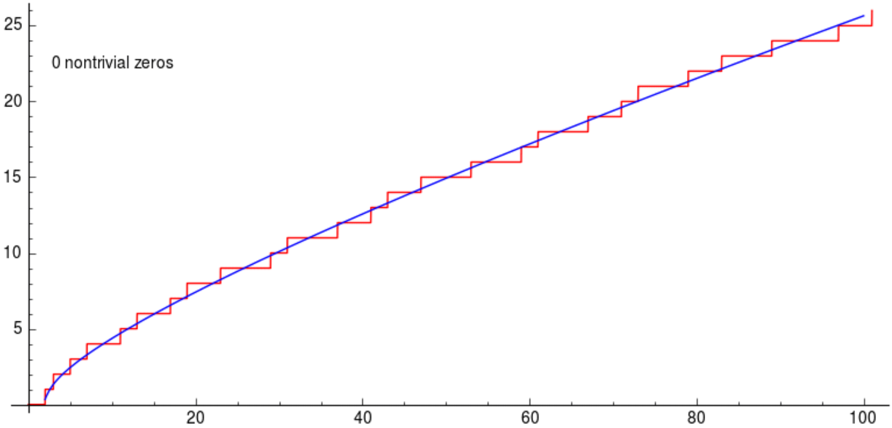
So finally we get

$$\pi(X) = R(X) - \sum_{\zeta(\rho)=0} R(X^\rho), \text{ where } R(X) = \sum_{n=1}^\infty \frac{\mu(n)}{n}\mathrm{li}(X^{\frac{1}{n}}).$$

The explicit formula in action



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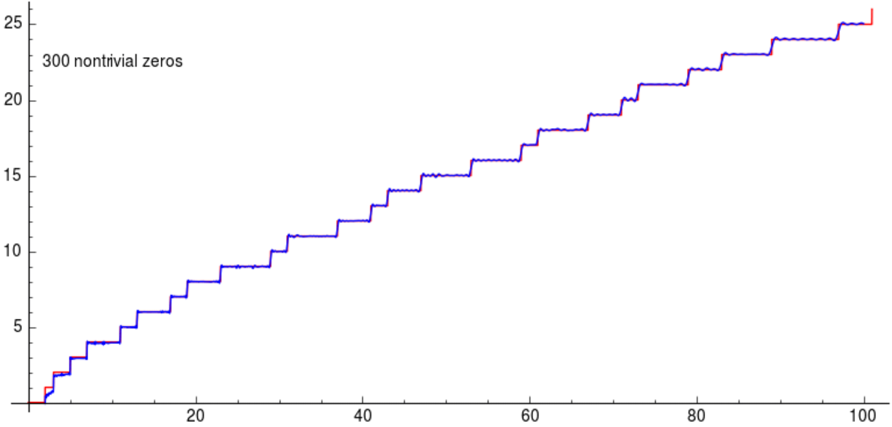

Zero-free regions (cont.)

The state-of-the-art is the Vinogradov-Korobov bound:

$$\zeta(\sigma + it) \neq 0$$

for

$$\sigma \geq 1 - \frac{c}{(\log|t| + 1)^{2/3}(\log\log(3 + |t|))^{1/3}}.$$

This bound doesn't even give a constant width.