Irrational and transcendental numbers: π, e, and others

LMS Summer School 2023

Lewis Combes

University of Sheffield

π

Everyone knows π is the ratio of the circumference of a circle to its diameter. Hopefully.

Everyone knows π is the ratio of the circumference of a circle to its diameter. Hopefully.

It is also the area of the unit circle.

π

Everyone knows π is the ratio of the circumference of a circle to its diameter. Hopefully.

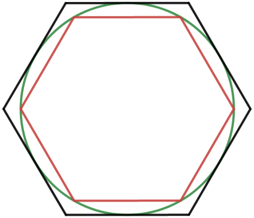
It is also the area of the unit circle.

First rigorous approach to approximate π used polygons. The idea: the area (or perimeter) of a polygon is easy* to find, so bound π by drawing polygons inside and outside the circle.

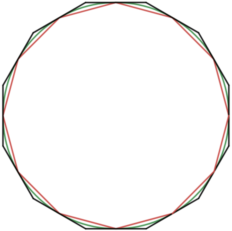
Everyone knows π is the ratio of the circumference of a circle to its diameter. Hopefully.

It is also the area of the unit circle.

First rigorous approach to approximate π used polygons. The idea: the area (or perimeter) of a polygon is easy* to find, so bound π by drawing polygons inside and outside the circle.



(g) 6 sides



(h) 12 sides

The polygon method

Archimedes used the polygon method to bound π

$$3.1408\ldots = \frac{223}{71} < \pi < \frac{22}{7} = 3.1428\ldots$$

by computing with a 96-gon. This process is *extremely slow*.

The polygon method

Archimedes used the polygon method to bound π

$$3.1408\ldots = \frac{223}{71} < \pi < \frac{22}{7} = 3.1428\ldots$$

by computing with a 96-gon. This process is *extremely slow*. It can be improved somewhat: Liu Hui developed a method that extracted the value

$$\pi \approx 3.1416$$

from a 96-gon.

The polygon method

Archimedes used the polygon method to bound π

$$3.1408\ldots = \frac{223}{71} < \pi < \frac{22}{7} = 3.1428\ldots$$

by computing with a 96-gon. This process is *extremely slow*. It can be improved somewhat: Liu Hui developed a method that extracted the value

$$\pi \approx 3.1416$$

from a 96-gon. This went on until the mid-1600s. Using a polygon with $2^{62} \approx 4.61 \times 10^{18}$ sides, Ludolph van Ceulen approximated π to 35 decimal places. It took 25 years.

The polygon method

Archimedes used the polygon method to bound π

$$3.1408\ldots = \frac{223}{71} < \pi < \frac{22}{7} = 3.1428\ldots$$

by computing with a 96-gon. This process is *extremely slow*. It can be improved somewhat: Liu Hui developed a method that extracted the value

$$\pi \approx 3.1416$$

from a 96-gon. This went on until the mid-1600s. Using a polygon with $2^{62} \approx 4.61 \times 10^{18}$ sides, Ludolph van Ceulen approximated π to 35 decimal places. It took 25 years. The last record set with this method was 20 years later, when 38 digits were obtained by Christoph Grienberger.

Irrational numbers

A **rational number** is a number that can be expressed as the ratio of two integers.

Irrational numbers

A **rational number** is a number that can be expressed as the ratio of two integers.

The rationals are written $\mathbb{Q}$. They sit inside the real numbers $\mathbb{R}$. The rationals are **countable** and the reals are **uncountable**, so most numbers are *not* rational.

Irrational numbers

A **rational number** is a number that can be expressed as the ratio of two integers.

The rationals are written $\mathbb{Q}$. They sit inside the real numbers $\mathbb{R}$. The rationals are **countable** and the reals are **uncountable**, so most numbers are *not* rational.

Examples of irrational numbers: π, e, $\sqrt{2}$, $\log(3)$, $\sqrt{2}^{\sqrt{2}}$, $\zeta(3)$,...

Irrational numbers

A **rational number** is a number that can be expressed as the ratio of two integers.

The rationals are written $\mathbb{Q}$. They sit inside the real numbers $\mathbb{R}$. The rationals are **countable** and the reals are **uncountable**, so most numbers are *not* rational.

Examples of irrational numbers: π, e, $\sqrt{2}$, $\log(3)$, $\sqrt{2}^{\sqrt{2}}$, $\zeta(3)$,...

So rational approximations of π are only ever going to be *approximations*.

Proof that $\sqrt{2}$ is irrational

Assume $\sqrt{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$.

Proof that $\sqrt{2}$ is irrational

Assume $\sqrt{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$.

Squaring and rearranging, we get

$$2b^2 = a^2 \rightsquigarrow a = 2k \text{ for some } m \in \mathbb{Z}$$

Proof that $\sqrt{2}$ is irrational

Assume $\sqrt{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$.

Squaring and rearranging, we get

$$2b^2 = a^2 \rightsquigarrow a = 2k \text{ for some } m \in \mathbb{Z}$$

Subbing it back in, we get $2b^2 = (2k)^2 = 4k^2$, and so $b^2 = 2k^2$. Once again, b and k are both integers, so

$$b = 2n \text{ for some } n \in \mathbb{Z}.$$

Proof that $\sqrt{2}$ is irrational

Assume $\sqrt{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$.

Squaring and rearranging, we get

$$2b^2 = a^2 \rightsquigarrow a = 2k \text{ for some } m \in \mathbb{Z}$$

Subbing it back in, we get $2b^2 = (2k)^2 = 4k^2$, and so $b^2 = 2k^2$. Once again, b and k are both integers, so

$$b = 2n \text{ for some } n \in \mathbb{Z}.$$

Subbing back in, we get $4n^2 = 2k^2$, so

$$4n^2 = 2k^2 \rightsquigarrow 2n^2 = k^2.$$

Proof that $\sqrt{2}$ is irrational

Assume $\sqrt{2} = \frac{a}{b}$, with $\gcd(a, b) = 1$.

Squaring and rearranging, we get

$$2b^2 = a^2 \rightsquigarrow a = 2k \text{ for some } m \in \mathbb{Z}$$

Subbing it back in, we get $2b^2 = (2k)^2 = 4k^2$, and so $b^2 = 2k^2$. Once again, b and k are both integers, so

$$b = 2n \text{ for some } n \in \mathbb{Z}.$$

Subbing back in, we get $4n^2 = 2k^2$, so

$$4n^2 = 2k^2 \rightsquigarrow 2n^2 = k^2.$$

We note that n and k are both integers, so...

Proving a number is irrational

A proof that a given number x is irrational is necessarily *by contradiction*. That contradiction is (almost) always that **there is a smallest natural number**.

Proving a number is irrational

A proof that a given number x is irrational is necessarily *by contradiction*. That contradiction is (almost) always that **there is a smallest natural number**.

Every proof is *bespoke*; every number we know is irrational is proved by a proof that uses its particular properties. This makes irrationality proofs *hard*.

Proving a number is irrational

A proof that a given number x is irrational is necessarily *by contradiction*. That contradiction is (almost) always that **there is a smallest natural number**.

Every proof is *bespoke*; every number we know is irrational is proved by a proof that uses its particular properties. This makes irrationality proofs *hard*.

Numbers we *think* are irrational but can't prove: $\pi \pm e$, πe π/e, π^e, $\arctan(\pi)$, γ (the Euler gamma constant), $\zeta(5)$, e^{e^e}, $\cos(\cos(1))$,...

Proving a number is irrational

A proof that a given number x is irrational is necessarily *by contradiction*. That contradiction is (almost) always that **there is a smallest natural number**.

Every proof is *bespoke*; every number we know is irrational is proved by a proof that uses its particular properties. This makes irrationality proofs *hard*.

Numbers we *think* are irrational but can't prove: $\pi \pm e$, πe π/e, π^e, $\arctan(\pi)$, γ (the Euler gamma constant), $\zeta(5)$, e^{e^e}, $\cos(\cos(1))$,...

The list goes on and on.

Continued fractions

The irrationality of π and e were both proven using **continued fractions**:

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

If a_n is *eventually always zero*, the fraction represents a rational number. If this never happens, the fraction is infinite represents an irrational number.

Continued fractions

The irrationality of π and e were both proven using **continued fractions**:

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

If a_n is *eventually always zero*, the fraction represents a rational number. If this never happens, the fraction is infinite represents an irrational number.

$$\frac{3}{5} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}, \quad \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) \approx x$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) \approx \frac{x}{1 - x^2}$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) \approx \cfrac{x}{1 - \cfrac{x^2}{3}}$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) \approx \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5}}}$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) \approx \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7}}}}$$

The first proof that π is irrational

It was Johann Lambert who gave the first proof that π is irrational.

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}$$

The first proof is due to Euler. It uses continued fractions.

This method is due to Fourier.

Algebraic numbers

The algebraic numbers form a set that generalise the rationals. A number θ is **algebraic** if it is the root of some rational polynomial, i.e.

$$a_n\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_1\theta + a_0 = 0,$$

with $a_0, \ldots, a_n \in \mathbb{Q}$.

Algebraic numbers

The algebraic numbers form a set that generalise the rationals. A number θ is **algebraic** if it is the root of some rational polynomial, i.e.

$$a_n\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_1\theta + a_0 = 0,$$

with $a_0, \ldots, a_n \in \mathbb{Q}$. Lots of familiar numbers are algebraic.

- $\sqrt{2} \rightsquigarrow x^2 - 2,$
- $\mathrm{i} = \sqrt{-1} \rightsquigarrow x^2 + 1,$
- $\cos(\frac{2\pi}{9}) \rightsquigarrow 8x^3 - 6x + 1,$
- $\ldots$

Algebraic numbers

The algebraic numbers form a set that generalise the rationals. A number θ is **algebraic** if it is the root of some rational polynomial, i.e.

$$a_n\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_1\theta + a_0 = 0,$$

with $a_0, \ldots, a_n \in \mathbb{Q}$. Lots of familiar numbers are algebraic.

- $\sqrt{2} \rightsquigarrow x^2 - 2$,
- $\mathrm{i} = \sqrt{-1} \rightsquigarrow x^2 + 1$,
- $\cos(\frac{2\pi}{9}) \rightsquigarrow 8x^3 - 6x + 1$,
- $\ldots$

The sum, product and quotient of two algebraic numbers is still algebraic, so they form a **field**.

Algebraic numbers

The algebraic numbers form a set that generalise the rationals. A number θ is **algebraic** if it is the root of some rational polynomial, i.e.

$$a_n\theta^n + a_{n-1}\theta^{n-1} + \ldots + a_1\theta + a_0 = 0,$$

with $a_0, \ldots, a_n \in \mathbb{Q}$. Lots of familiar numbers are algebraic.

- $\sqrt{2} \;\rightsquigarrow\; x^2 - 2$,
- $\mathrm{i} = \sqrt{-1} \;\rightsquigarrow\; x^2 + 1$,
- $\cos(\frac{2\pi}{9}) \;\rightsquigarrow\; 8x^3 - 6x + 1$,
- $\ldots$

The sum, product and quotient of two algebraic numbers is still algebraic, so they form a **field**.

So we know that $\frac{\sqrt{2}+\mathrm{i}}{\cos(\frac{2\pi}{9})}$ is algebraic, even if we can't immediately "spot" its polynomial.

Transcendental numbers

Note: the roots of a polynomial with *algebraic* coefficients are **also** algebraic numbers!

e.g. roots of $x^2 - \sqrt{2}x + 1$ are also roots of $x^4 + 1$.

Transcendental numbers

Note: the roots of a polynomial with *algebraic* coefficients are **also** algebraic numbers!

e.g. roots of $x^2 - \sqrt{2}x + 1$ are also roots of $x^4 + 1$.

The algebraic numbers form a countable subset of the *complex* numbers $\mathbb{C}$, so "most" numbers are *not* algebraic. Non-algebraic numbers are called **transcendental**.

Examples of transcendental numbers: π, e, $\log(2)$, $e^{\pi\sqrt{163}}$,...

Transcendental numbers

Note: the roots of a polynomial with *algebraic* coefficients are **also** algebraic numbers!

e.g. roots of $x^2 - \sqrt{2}x + 1$ are also roots of $x^4 + 1$.

The algebraic numbers form a countable subset of the *complex* numbers $\mathbb{C}$, so "most" numbers are *not* algebraic. Non-algebraic numbers are called **transcendental**.

Examples of transcendental numbers: π, e, $\log(2)$, $e^{\pi\sqrt{163}}$,...

Theorem (Lindemann-Weierstrass)

Suppose $\theta \neq 0$ is algebraic. Then e^θ is transcendental.

Long complicated proof with lots of technical detail. Once it's done, it is easy to prove transcendence of lots of numbers.

Quick proofs of transcendence

Theorem (Lindemann-Weierstrass)

Suppose $\theta \neq 0$ is algebraic. Then e^θ is transcendental.

Theorem

e is transcendental.

Quick proofs of transcendence

Theorem (Lindemann-Weierstrass)

Suppose $\theta \neq 0$ is algebraic. Then e^{θ} is transcendental.

Theorem

e is transcendental.

Proof.

Use Lindemann-Weierstrass with $\theta = 1$. $\square$

Quick proofs of transcendence

Theorem (Lindemann-Weierstrass)

Suppose $\theta \neq 0$ is algebraic. Then e^{θ} is transcendental.

Theorem

e is transcendental.

Proof.

Use Lindemann-Weierstrass with $\theta = 1$. $\square$

Theorem

π is transcendental.

Quick proofs of transcendence

Theorem (Lindemann-Weierstrass)

Suppose $\theta \neq 0$ is algebraic. Then e^{θ} is transcendental.

Theorem

e is transcendental.

Proof.

Use Lindemann-Weierstrass with $\theta = 1$. $\square$

Theorem

π is transcendental.

Proof.

Use Lindemann-Weierstrass with $\theta = \mathrm{i}\pi$. $\square$

Guiding principle

As noted, "most" numbers are transcendental. Numbers like e, π arose naturally in the history of mathematics and turned out to be transcendental.

We expect $e + \pi$ to be transcendental too. The sum of two transcendental numbers need not be transcendental, e.g. $\pi + (-\pi) = 0$. These two are transcendental for "the same reason". Nobody thinks e and π are transcendental for "the same reason". But of course, nobody knows.

A general principle in maths: if a number has been written down, studied, and isn't obviously algebraic, it is probably transcendental.

An algebraic surprise

We will play a game with integer sequences. Observe the following sequence:

$$\begin{array}{r} 1 \\ 11 \\ 21 \\ 1211 \\ 111221 \\ 312211 \\ \vdots \end{array}$$

Can you see the next number?

An algebraic surprise (cont.)

The sequence $1, 11, 21, 1211, 111221, 312211, \ldots$ is the **Conway look-and-say** sequence. The next number is 13112221.

The sequence $1, 11, 21, 1211, 111221, 312211, \ldots$ is the **Conway look-and-say** sequence. The next number is 13112221.

Write L_n for the length of the n^{th} term. Then the number

$$\lambda := \frac{L_{n+1}}{L_n} \approx 1.303577269\ldots$$

An algebraic surprise (cont.)

The sequence $1, 11, 21, 1211, 111221, 312211, \ldots$ is the **Conway look-and-say** sequence. The next number is 13112221.

Write L_n for the length of the n^{th} term. Then the number

$$\lambda := \frac{L_{n+1}}{L_n} \approx 1.303577269\ldots$$

is **algebraic**, satisfying a polynomial of degree 71.

An algebraic surprise (cont.)

$$
\begin{aligned}
&-6+3x-6x^2+12x^3-4x^4+7x^5-7x^6+x^7+5x^9-2x^{10}\\
&-4x^{11}-12x^{12}+2x^{13}+7x^{14}+12x^{15}-7x^{16}-10x^{17}-4x^{18}\\
&+3x^{19}+9x^{20}-7x^{21}-8x^{23}+14x^{24}-3x^{25}+9x^{26}+2x^{27}\\
&-3x^{28}-10x^{29}-2x^{30}-6x^{31}+x^{32}+10x^{33}-3x^{34}+x^{35}+7x^{36}\\
&-7x^{37}+7x^{38}-12x^{39}-5x^{40}+8x^{41}+6x^{42}+10x^{43}-8x^{44}-8x^{45}\\
&-7x^{46}-3x^{47}+9x^{48}+x^{49}+6x^{50}+6x^{51}-2x^{52}-3x^{53}-10x^{54}\\
&-2x^{55}+3x^{56}+5x^{57}+2x^{58}-x^{59}-x^{60}-x^{61}-x^{62}-x^{63}+x^{64}\\
&+2x^{65}+2x^{66}-x^{67}-2x^{68}-x^{69}+x^{71}
\end{aligned}
$$