Elliptic curves

LMS Summer School 2023

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Diophantine equations

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Examples:

- $x^2 + y^2 = z^2$
- $x^3 + y^3 = z^3$
- $x^2 - y^3 = 1$
- $x^3 + (x+1)^3 + (x+2)^3 = y^3$

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The integral solutions to this equation are well-understood. There are infinitely many solutions, or none, depending on the gcd of a, b and whether it divides c.

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After linear equations, come quadratic equations.

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Finding the rational solutions to these equations is a solved problem, and uses the *p-adic numbers*.

What is an elliptic curve?

Going up to degree 3, we get equations of the form:

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We can use linear transformations to put this equation in a "standard form":

$$E : y^2 = x^3 + Ax + B,$$

for some $A, B \in \mathbb{Q}$. This is an **elliptic curve**.

Points on elliptic curves

$$E : y^2 = x^3 - 4x + 4$$

Points on E:

$$\begin{aligned}(x, y) &= (-1, \sqrt{7}) \\ &= (-5, \sqrt{-101}) \\ &= \left(\frac{\sqrt[3]{18 - 2\sqrt{33}}}{\sqrt[3]{3}^2} - \frac{\sqrt[3]{2}^5}{\sqrt[3]{27 - 3\sqrt{33}}}, 0 \right) \\ &= (310, 5458)\end{aligned}$$

Rational points on elliptic curves

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Example:

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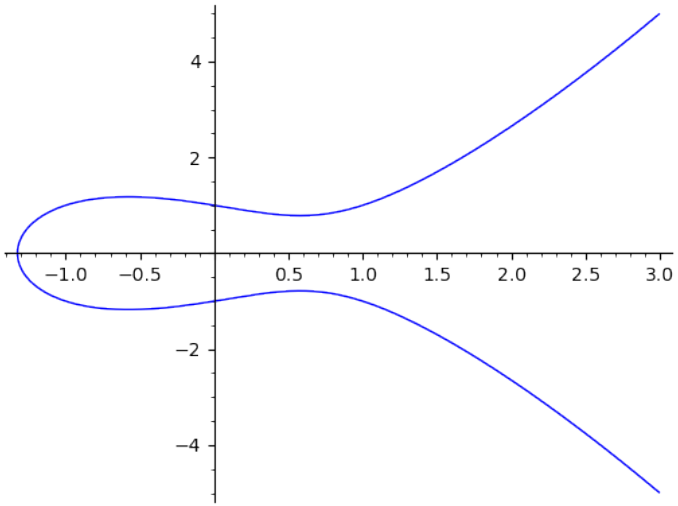
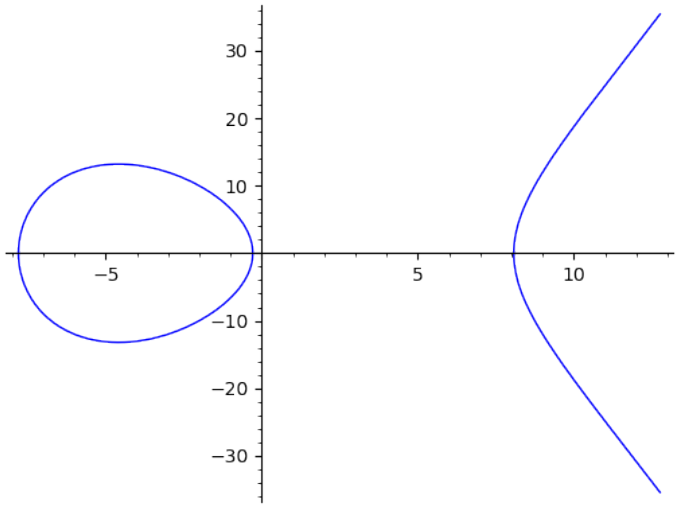
There is an algorithm to figure out the exact number of points that always works. Nobody knows if it *does* always work. If you can prove it does, you can claim a \$1,000,000 prize from the Clay Institute.



(a) $y^2 = x^3 - x + 1$



(b) $y^2 = x^3 - 63x - 18$

Figure: Elliptic curve pictures borrowed from the LMFDB.

The first elliptic curve

Diophantus of Alexandria wrote *Arithmetica* around 200AD. It consisted of many problems that we would recognise as Diophantine equations.

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Diophantus went on to find solutions in the case of $a = 6$:

$$6Y - Y^2 = X^3 - X.$$

Can you spot any easy ones?

The first elliptic curve (cont.)

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A non-obvious solution: $(X, Y) = \left(-\frac{10370209823}{214448643396}, -\frac{797444260812577}{99308164475680056}\right)$.

The first elliptic curve (cont.)

Clearly something mysterious going on. Diophantus himself produced the solution

$$(X, Y) = \left(\tfrac{17}{9}, \tfrac{26}{27}\right).$$

He does so using **the group law** on the elliptic curve.

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A **group** is a set G with a binary operation $\cdot$ such that

1. For all $g, h \in G$, one has $g \cdot h \in G$
2. There is a distinguished element Id_G such that $g \cdot \mathrm{Id}_G = \mathrm{Id}_G \cdot g = g$ for all $g \in G$.
3. For every $g \in G$, there is a $g^{-1} \in G$ such that $g \cdot g^{-1} = \mathrm{Id}_G$.
4. The operation $\cdot$ is **associative**—i.e. $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

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The group law on elliptic curves

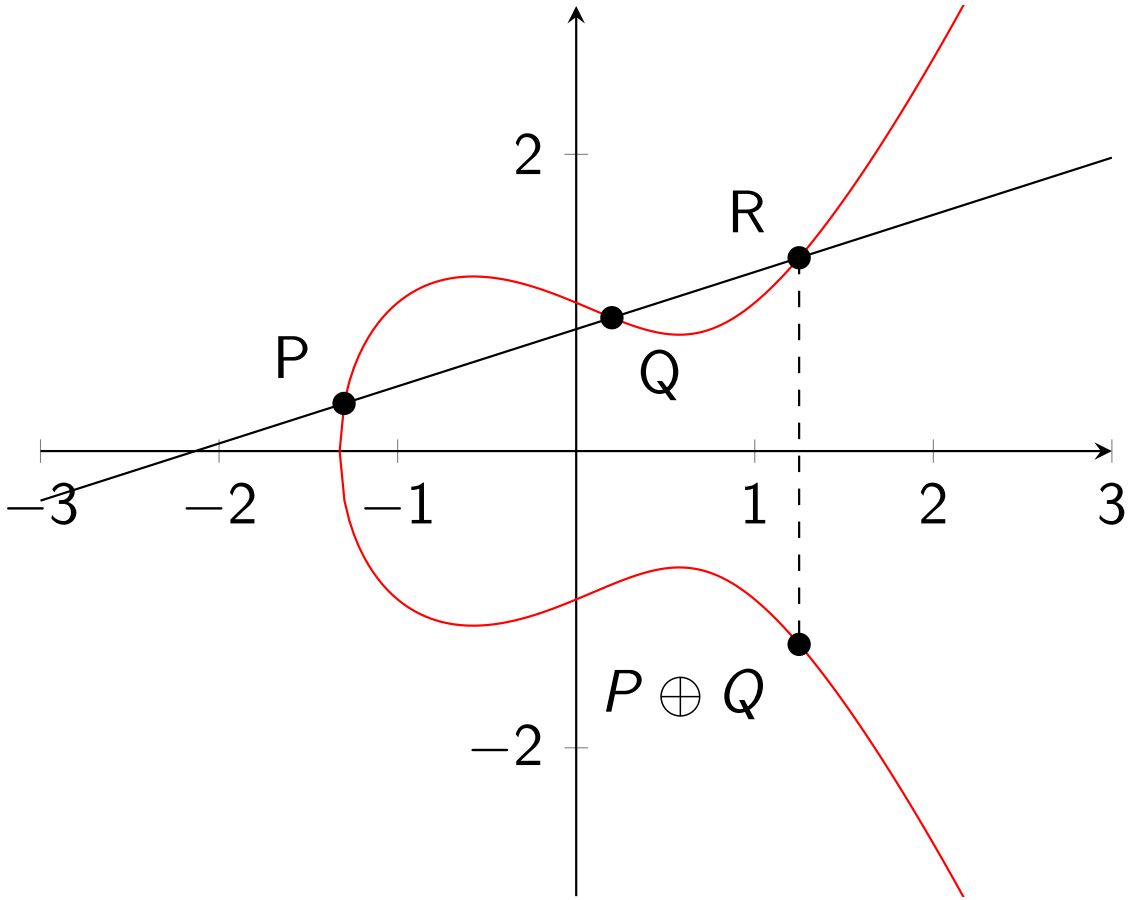

Figure: Elliptic Curve addition.

The group law on elliptic curves (cont.)

Diophantus used the group law (though he did not know it) to produce his solution $(X, Y) = \left(\frac{17}{9}, \frac{26}{27}\right)$ to $E : 6Y - Y^2 = X^3 - X$.

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It is a long and tedious calculation.

```
> E;
Elliptic Curve defined by y^2 - 6*y = x^3 - x over Rational Field
> P:=RationalPoints(E : Bound:=50)[6];
> P;
(1 : 6 : 1)
> time P+P;
(-17/9 : 26/27 : 1)
Time: 0.000
>
```

So what?

The group law on elliptic curves is **special**. A random curve is very unlikely to have a group law. This makes studying elliptic curves somewhat feasible, relative to curves defined by polynomials of higher degree.

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Points on an elliptic curve E have the structure of a **finitely-generated abelian group**. The group of rational points, written $E(\mathbb{Q})$, takes the special form

$$E(\mathbb{Q}) \simeq T + \mathbb{Z}^r.$$

Two parts:

- T: finite subgroup of **torsion points**,
- r: the number of independent generators of infinite order, called the **rank.**

The mysterious rank

The torsion subgroup T is well-understood. In fact, it has been classified exactly. There are only 15 possible groups T can be.

The rank is much more mysterious.

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$$y^2 = x^3 + x + 29 \rightsquigarrow r = 0 \ \& \ T = C_1$$
$$y^2 = x^3 + x + 30 \rightsquigarrow r = 0 \ \& \ T = C_2$$
$$y^2 = x^3 + x + 31 \rightsquigarrow r = 1 \ \& \ T = C_1$$

Birch and Swinnerton-Dyer's conjecture

In the year 2000, the Clay Institute issued a list of seven **Millenium Prize Problems**. The Birch and Swinnerton-Dyer conjecture is one of these problems.

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Fundamental idea: look at the elliptic curves modulo prime numbers.

It is **not enough** to find points on a curve mod p and lift them to $\mathbb{Q}$.

Birch and Swinnerton Dyer's conjecture (cont.)

However, it is still worth a try.

Principle: if an elliptic curve has rank > 0, it has "lots of points", so it should still have "lots of points" $(\mathrm{mod}\ p)$.

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Birch and Swinnerton-Dyer developed the conjecture in the 1960s.

(c) Birch and Swinnerton-Dyer

(d) Computers in the 1960s

Elliptic curves over $\mathbb{F}_p$

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

$$E : y^2 = x^3 - x + 9$$

Let $p = 5$. A **point on E over $\mathbb{F}_5$** is a pair $(x, y) \in \mathbb{F}_p^2$ satisfying E.

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E.g. $(x, y) = (4, 2)$.

Hasse-Weil bound tell us

$$-2\sqrt{p} + p + 1 \leq \#E(\mathbb{F}_p) \leq 2\sqrt{p} + p + 1.$$

or, if you prefer,

$$|\#E(\mathbb{F}_p) - (p + 1)| \leq 2\sqrt{p}.$$

BSD example

The original statement of the conjecture is the following:

$$\prod_{p \leq X} \frac{\#E(\mathbb{F}_p)}{p} \approx C \log(X)^{\mathrm{rank}(E)}$$

as $X \to \infty$. Here C is some constant.

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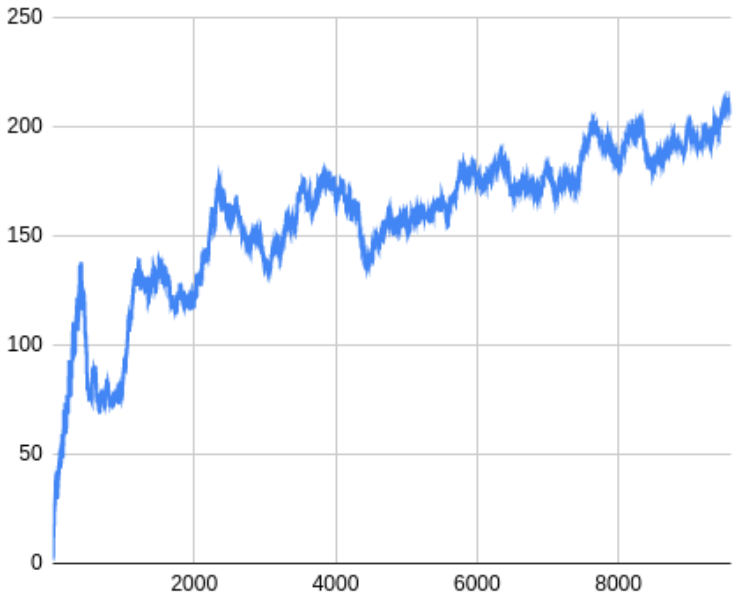
E.g. $E_1 : y^2 = x^3 - x + 9$, and $E_2 : y^2 = x^3 - x + 5$.

p	2	3	5	7	11	13	17	19	23	29	31	37
$\#E_1(\mathbb{F}_p)$	3	4	8	9	16	15	24	16	32	36	37	39
$\#E_2(\mathbb{F}_p)$	3	1	8	7	13	10	17	20	18	36	30	32

Which one of these curves "should" have rank > 0?

BSD example (cont.)



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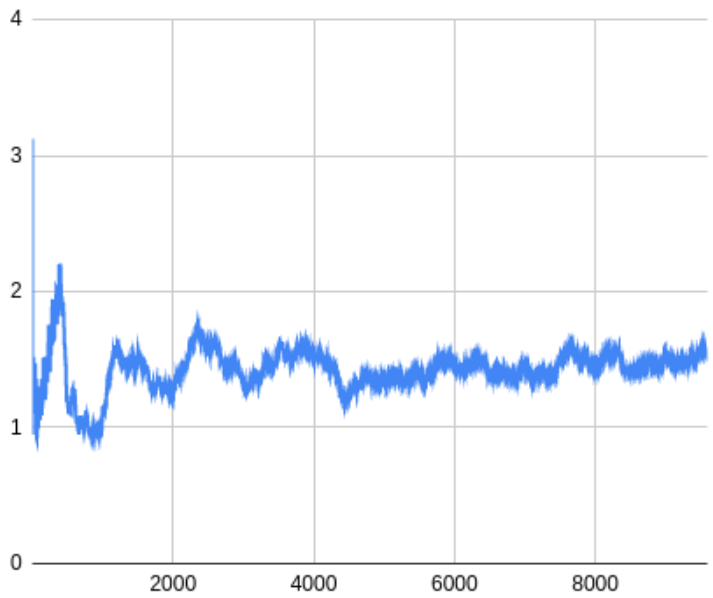

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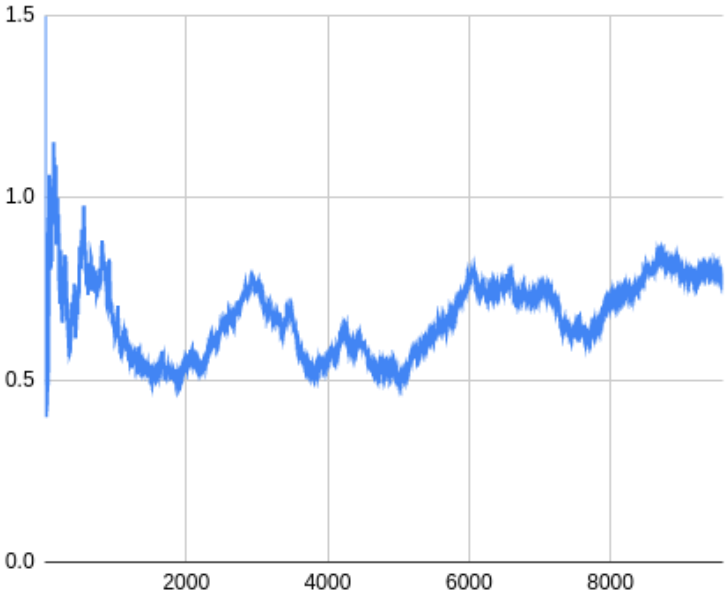
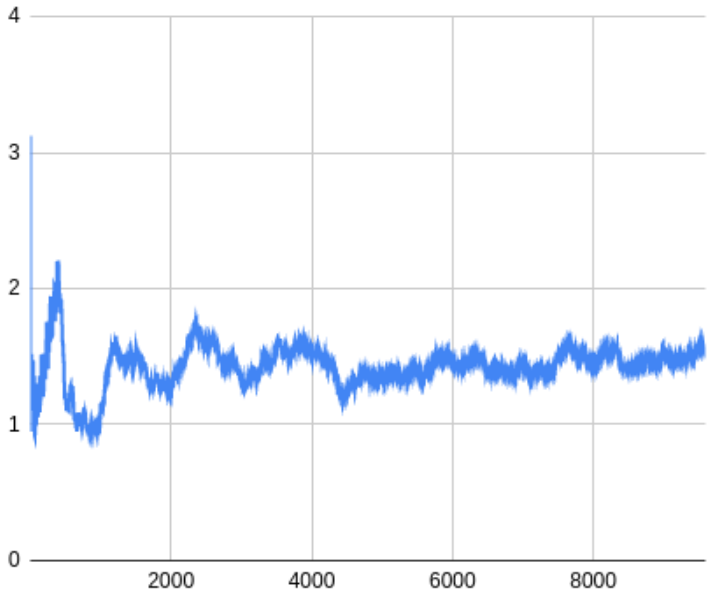


(k) $\frac{1}{\log(X)^2} \prod_{n \leq X} \frac{\#E_1(\mathbb{F}_p)}{p}$



(l) $\prod_{n \leq X} \frac{\#E_2(\mathbb{F}_p)}{p}$

Current state of BSD

Combining the work of Coates & Wiles, Gross & Zagier, Kolyvagin, Wiles, Taylor & Wiles and Breuil-Conrad-Diamond-Taylor, one has the following:

Theorem (due to everyone above + more)

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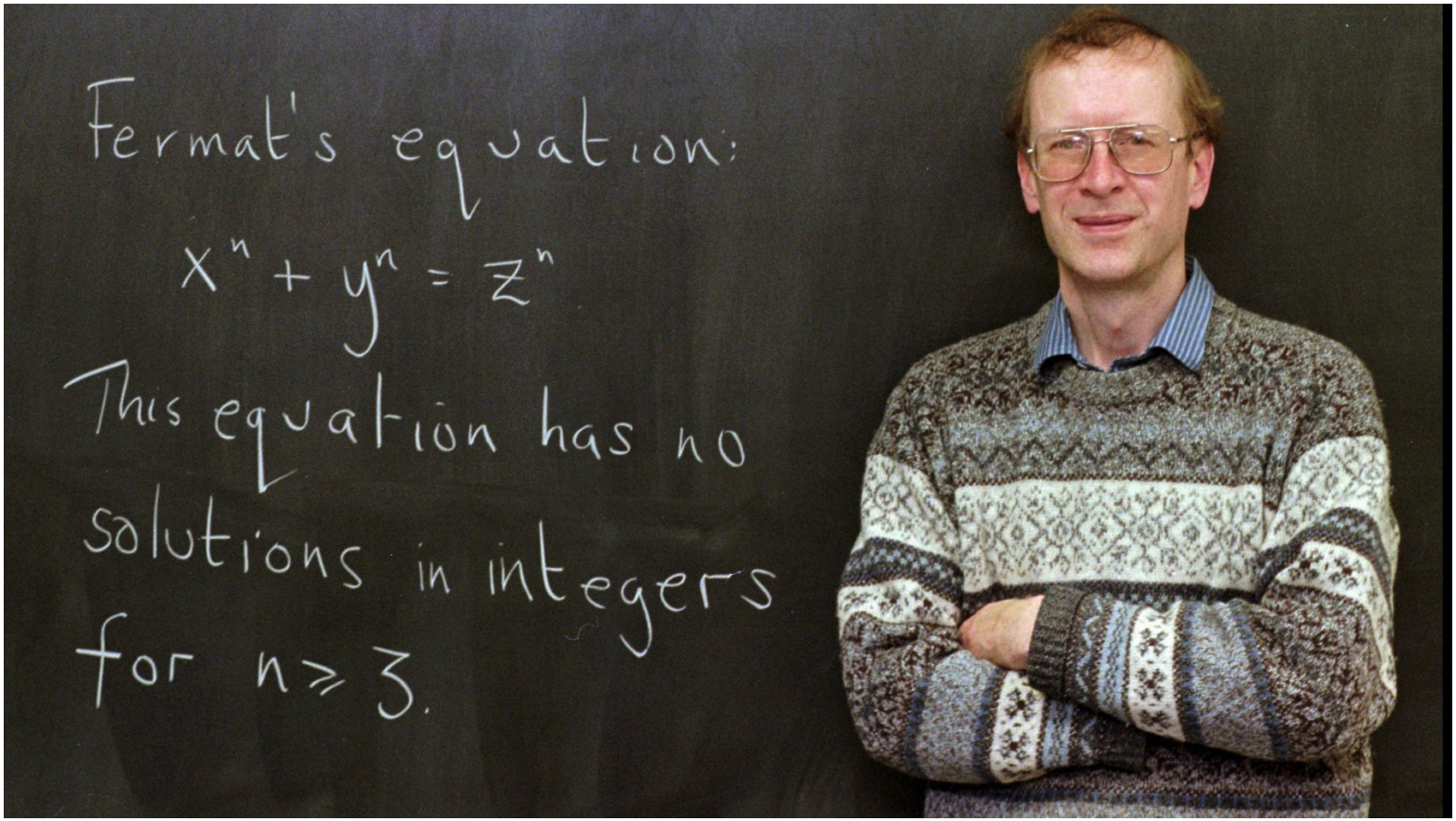
The Clay Institute's \$1,000,000 is still waiting to be claimed...

Fermat's Last Theorem



Fermat's Last Theorem (cont.)

Wiles' strategy: start with an assumed solution $a^p + b^p = c^p$ for a prime p. Use this to create an elliptic curve

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Because $a, b, c \in \mathbb{Z}$, the curve E ends up having very special properties.

In particular, it cannot be **modular**. Wiles proved that all* elliptic curves *are* modular, so the solution leads to a curve that can't exist.